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Regularity of Weak Solutions of the Compressible Navier-Stokes Equations

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Abstract

We prove regularity of weak solutions of the Navier-Stokes equations for compressible, isentropic flow in three space dimension. We allow the presence of vacuum region for the initial data. The pressure law satisfies the general relation $P(\rho)=a\rho^{\gamma},\gamma\geq 1$. As was found by Hoff[2], Lions[7] and Desjardins[1], the effective viscosity G plays an important role.

keywords: Navier-Stokes equations, isentropic, weak solution, regularity

1 Introduction

The isothermal gases are governed by isentropic compressible Navier-Stokes equations. Although there are many important results, the existence of solutions under general condition remains still open. When the initial velocity has small norm in sufficiently regular space, say H^3, and the initial density is near constant, the global existence of classical solution was obtained by Matsumura and Nishida[9]. Then, Hoff[2] extended the global existence of small solutions to more weaker spaces which allow discontinuity of the initial

For the weak solutions, Lions[7] obtained the global existence when the pressure law satisfies $P(\rho) = a\rho^\gamma, \gamma \geq 9/5$ for three space dimension, $\gamma \geq 3/2$ for two space dimension and $\gamma > N/2$ for N-space dimension with $N \geq 4$. Now, the remaining question will be the extension of the range of the parameter γ.

On the contrary, Solonnikov[13] showed the local existence of strong solutions if there is no vacuum region for the initial density in the context of classical. Also, Desjardins[1] proved local regularity for the weak solutions when $\gamma \geq 1$ for two space dimension and $\gamma > 3$ for three space dimension.

In this paper, we prove the a priori regularity of weak solutions under the general law $P(\rho) = a\rho^\gamma, \gamma \geq 1$ for three space dimension. We allow vacuum region and do not assume any smallness for the initial data. The compactness and local existence of strong solution will be discussed in a forthcoming paper.

First, we consider the isentropic compressible Navier-Stokes equations in periodic domain $\mathbf{T}^3$ with periodicity one to each coordinate direction:

$$\rho_t + \text{div}(\rho u) = 0 \quad \text{in} \quad (0,T) \times \mathbf{T}^3$$

$$(\rho u)_t + \text{div}(\rho u \otimes u) - \mu \Delta u - (\lambda + \mu)\nabla \text{div}(u) + \nabla P(\rho) = \rho f \quad \text{in} \quad (0,T) \times \mathbf{T}^3,$$

where the pressure satisfies for a positive constant a

$$P(\rho) = a\rho^\gamma, \quad \gamma \geq 1.$$

The viscosity constants satisfy $\mu > 0$ and $\lambda + \mu \geq 0$ and the external force f belongs to $L^2((0,T) \times \mathbf{T}^3)$. We need to find the unknown velocity $u \in \mathbf{R}^3$ and the unknown density $\rho \in \mathbf{R}$. The velocity and pressure are to satisfy the initial condition

$$\rho(0,x) = \rho_0(x), u(0,x) = u_0(x).$$

Although we do not know yet the global existence of weak solution under the general pressure law, we introduce definition of a weak solution. In fact the estimates of local smoothness of the weak solution will lead to the existence of strong solution and we will discuss the existence in different places. $(\rho, u) \in L^1((0,T_0) \times \mathbf{T}^3)$ is a weak solution if it satisfies

$$\int \rho_0 \psi(0,x)dx + \int_0^{T_0} \int \rho \psi_t + \rho u \cdot \nabla \psi dxdt = 0$$

$$\int \rho_0 u_0 \psi(x,0)dx + \int_0^{T_0} \int \rho u \otimes u \nabla u + P \text{div} \psi dxdt$$

$$= \int_0^{T_0} \int \mu \nabla u \nabla \psi + (\lambda + \mu) \mathrm{div} u \mathrm{div} \psi dx dt + \int_0^{T_0} \int \rho f \psi dx dt$$

for all $\psi \in C_0^\infty[0, T_0 : C^\infty(\mathbf{T}^3))$ which is periodic. Moreover (ρ, u) satisfies

$$\sup_{0 \le t \le T_0} |\rho|_\gamma(t) + |\sqrt{\rho} u|_2(t) + \int_0^{T_0} |\nabla u|_2 dt \le C.$$

We denote $|u|_p = (\int |u|^p dx)^{1/p}$ and c is constant depending only exterior data.

Theorem 1.1 *Suppose that $\rho_0 \in L^\infty$ and $u_0 \in H^1$. Then, there is T such that the weak solution (ρ, u) satisfies $\rho \in L^\infty([0, T) \times \mathbf{T}^3)$ and $u \in L^\infty(0, T : H^1(\mathbf{T}^3))$. Furthermore we have*

$$\sup_{0 \le t \le T} |\rho|_\infty(t) + |\nabla u|_2(t) + \left(\int_0^T |\sqrt{\rho} u_t|_2^2(t) dt \right)^{1/2} \le c.$$

For our simplicity of presentation, we assume zero external force.

2 Estimate of integral norm of density

We define our objective function h by

$$h(t) = |\rho|_\infty(t) + |\nabla u|_2(t).$$

For computational convenience we introduce two universal Lipschitz function Φ and Ψ which could be different in each appearance. $\Phi(h(s))$ depends only on $h(s)$ and $\Psi(\int_0^t \Phi ds)$ depends only on $\int_0^t \Phi(h(s)) ds$ But, after overall computations, they will be decided in natural way.

First, we estimate the Averages. We denote $\overline{u} = \int u dx$. The initial mass is positive so that

$$\int \rho_0 dx = M > 0,$$

otherwise the problem is trivial. From mass conservation and momentum conservation,

$$\overline{\rho}(t) = M \quad \text{and} \quad \overline{\rho u}(t) = \int \rho_0 u_0 dx$$

for all t. From Poincaré inequality we have

$$|\int \rho(u - \overline{u}) dx(t)| \le |\rho|_\infty \left(\int |u - \overline{u}|^2 dx \right)^{1/2}$$

$$\leq c|\rho|_\infty |\nabla u|_2(t)$$

and hence we obtain

$$|\overline{u}|(t) \leq \frac{1}{M}\left|\int \rho u dx(t)\right| + \frac{|\rho|_\infty(t)}{M}|\nabla u|_2(t) \leq \Phi(h(t))$$

for some Lipschitz function Φ. We also have

$$\left|\int \rho |u|^2 dx(t) - M\overline{|u|^2}(t)\right| \leq \int \rho \left||u|^2 - \overline{|u|^2}\right| dx(t)$$

$$\leq |\rho|_\infty(t) \int |u||\nabla u| dx(t) \leq \frac{1}{4} M\overline{|u|^2}(t) + 4\left(\frac{|\rho|_\infty(t)}{M}\right)^2 |\nabla u|_2^2(t)$$

and hence it follows that

$$\overline{|u|^2}(t) \leq \frac{2}{M}\int \rho|u|^2 + 6\left(\frac{|\rho|_\infty(t)}{M}\right)^2 |\nabla u|_2^2(t) \leq \Phi(h(t)).$$

Now we estimate the integral norms of density. We apply ρ^{k-1} as a test function to mass conservation. Then, we obtain

$$(\rho^k)_t + \mathrm{div}(\rho^k u) + (k-1)\rho^k \mathrm{div}(u) = 0$$

for any positive constant k and hence integrating in time and space

$$\int \rho^k dx(t) = \int \rho_0^k dx - (k-1)\int_0^t \int \rho^k(s,x)\mathrm{div}(u(s,x)) dx ds$$

$$\leq \int \rho_0^k dx + \int_0^t |\nabla u|_2^2(s) ds + c(k)|\rho|_\infty^{2k}(s) ds$$

$$\leq c + \int_0^t \Phi(h(s)) ds.$$

Therefore we conclude

$$|\rho|_k(t) \leq \Psi\left(\int_0^t \Phi(h(s)) ds\right)$$

for all fixed positive constant and for some Lipschitz functions Φ and Ψ. We decide appropriate k later.

3 Estimate of velocity

To handle the nonlinear convection term $\rho u\cdot\nabla u$, we first estimate

$$\sup_{0\le s\le t}\int \rho|u|^4dx(s)+\int_0^t\int |u|^2|\nabla u|^2dxds.$$

For our convenience we define effective pressure Q and effective viscosity flux G by

$$Q=-(\lambda+\mu)\mathrm{div}(u)+P(\rho)$$
$$G=(\lambda+2\mu)\mathrm{div}(u)-P(\rho)=\mu\mathrm{div}(u)-Q.$$

Taking $|u|^2u$ as test function for momentum conservation equation, we have

$$\frac{1}{4}\int \rho(|u|^4)_tdx+\frac{1}{4}\int\rho u\cdot\nabla(|u|^4)dx+\mu\int|u|^2|\nabla u|^2dx$$
$$+\frac{\mu}{8}\int|\nabla(|u|^2)|^2dx=\int Q\mathrm{div}(|u|^2u)dx.$$

We note that

$$\int\rho(|u|^4)_tdx+\int\rho u\cdot\nabla(|u|^4)dx=\frac{d}{dt}\int\rho|u|^4dx.$$

Hence, integrating in time, we have

$$\int\rho|u|^4dx(t)+\mu\int_0^t\int|u|^2|\nabla u|^2dxds$$
$$\le\int\rho_0|u_0|^4dx+c\int_0^t\int|Q||u||u\nabla u|dxds.$$

It is important to find right exponent to derive closed estimates. From Hölder inequality and Sobolev inequality, we have

$$\int_0^t\int|Q||u||u\nabla u|dxds\le\int_0^t\left[\int|Q|^{12/5}dx\right]^{5/12}\left[\int(|u|^2)^6dx\right]^{1/12}\left[\int|u|^2|\nabla u|^2dx\right]^{1/2}ds$$
$$\le\int_0^t\left[\int|Q|^{12/5}dx\right]^{5/12}\left[\int(|u|^2-\overline{|u|^2}(s))^6dx\right]^{1/12}\left[\int|u|^2|\nabla u|^2dx\right]^{1/2}ds$$
$$+\int_0^t(\overline{|u|^2})^{1/2}\left[\int|Q|^{12/5}dx\right]^{5/12}\left[\int|u|^2|\nabla u|^2dx\right]^{1/2}ds$$

$$\leq c \int_0^t \left[\int |Q|^{12/5} dx \right]^{5/12} \left[\int |u|^2 |\nabla u|^2 dx \right]^{3/4} ds$$

$$+ \int_0^t (\overline{|u|^2})^{1/2} \left[\int |Q|^{12/5} dx \right]^{5/12} \left[\int |u|^2 |\nabla u|^2 dx \right]^{1/2} ds$$

$$\leq c \int_0^t \left[\int |Q|^{12/5} dx \right]^{5/3} + \overline{|u|^2} \left[\int |Q|^{12/5} dx \right]^{5/6} ds$$

$$+ \frac{\mu}{4} \int_0^t \int |u|^2 |\nabla u|^2 dx ds.$$

The estimates for generalized pressure Q can be replaced by effective viscosity flux G so that

$$\int |Q(s,x)|^{12/5} dx \leq c \int |G(s,x)|^{12/5} dx + \Phi(h(s)).$$

From the definition of control variable h and G, we also have

$$|\overline{G}(s)| \leq \Phi(h(s)).$$

Thus from Sobolev inequality and, we find that

$$\left(\int |G(s,x)|^{12/5} dx \right)^{5/3} \leq \left(\int |G(s,x) - \overline{G}(s)|^{12/5} dx \right)^{5/3} + \Phi(h(s))$$

$$\leq \left(\int |G(s,x) - \overline{G}(s)|^2 dx \right)^{5/4} \left(\int |G(s,x) - \overline{G}(s)|^{18/5} dx \right)^{5/12} + \Phi(h(s))$$

$$\leq \varepsilon_0 \left(\int |G(s,x) - \overline{G}(s)|^{18/5} dx \right)^{1/2} + c \left(\int |G(s,x) - \overline{G}(s)|^2 dx \right)^{15/2} + \Phi(h(s)).$$

We note that

$$c \left(\int |G(s,x) - \overline{G}(s)|^2 dx \right)^{15/2} \leq c |\nabla u|_2^{15}(s) + |P(\rho)|_2^1 5 \leq \Phi(h(s))$$

and

$$\left(\int |G(s,x) - \overline{G}(s)|^{18/5} dx \right)^{1/2} \leq c |\nabla G|_{15/8}^{9/5}.$$

Here important fact is the exponent $9/5$ is less than 2 and $15/8$ is less also less than 2. Therefore combining all the previous estimates, we conclude

$$\sup_{0 \leq s \leq t} \int \rho |u|^4 dx(s) + \int_0^t \int |u|^2 |\nabla u|^2 dx ds$$

$$\leq\leq \left(\int_0^t |\nabla G|_{15/8}^2(s)ds\right)^{9/10} + \int_0^t \Phi(h(s))ds.$$

We let $\mathbf{P}$ be the projection operator to divergence free vector space. Then, from the definition of G and $\mathbf{P}u$, we have

$$\Delta G = \mathrm{div}(\rho u_t) + \mathrm{div}(\rho u \cdot \nabla u)$$

$$\Delta \mathbf{P}u = \mathbf{P}(\rho u_t + \rho u \cdot \nabla u).$$

For a given nonnegative constant $\delta \in [0,1)$, we have

$$|\nabla G|_{2-\delta}^2 + |\Delta \mathbf{P}u|_{2-\delta}^2 \leq c\left(|\rho u_t|_{2-\delta}^2 + |\rho u \cdot \nabla u|_{2-\delta}^2\right)$$

$$\leq c(|\rho|_m^2 + 1)\left(|\sqrt{\rho} u_t|_2^2 + |u \nabla u|_2^2\right)$$

for some m depends only on δ and integrating with respect to time we obtain

$$\int_0^t |\nabla G|_{2-\delta}^2 + |\Delta \mathbf{P}u|_{2-\delta}^2 ds$$

$$\leq c \sup_{0\leq s\leq t}(|\rho|_m^2 + 1)\int_0^t\int \rho|u_t|^2 + |u\nabla u|^2 dxds$$

$$\leq \Psi(\int_0^t \Phi(h(s))ds)\int_0^t\int \rho|u_t|^2 + |u\nabla u|^2 dxds.$$

Moreover, the Sobolev inequality implies that

$$|\nabla u|_5 \leq c|\nabla G|_{15/8} + c|\Delta \mathbf{P}u|_{15/8} + \Phi(h(s)).$$

Finally we estimate $|\nabla u|_2(t)$. We multiply u_t to our momentum conservation equation and integrate. Consequently, we have

$$\int_0^t\int \rho|u_t|^2 dxds + \mu\int |\nabla u|^2 dx(t) + (\lambda+\mu)\int |\mathrm{div} u|^2 dx(t)$$

$$\leq \mu\int |\nabla u_0|^2 dx + (\lambda+\mu)\int |\mathrm{div} u_0|^2 dx + \int_0^t\int \rho|u\nabla u|^2 dxds - \int_0^t\int \nabla p\cdot u_t dxds.$$

Again from Hölder inequality, for a given constant $0 < \varepsilon < 1$, we have

$$\int_0^t\int \rho|u\nabla u|^2 dxds \leq \left(\int_0^t\int |u\nabla u|^2 dxds\right)^{1-\varepsilon}\left(\int_0^t\int \rho^{1/\varepsilon}|u\nabla u|^2 dxds\right)^{\varepsilon}$$

$$\int_0^t \int \rho^{1/\varepsilon} |u\nabla u|^2 dxds \le \int_0^t \left(\int \rho^{10/\varepsilon - 5} dx \right)^{1/10} \left(\int \rho |u|^4 dx \right)^{1/2} \left(\int |\nabla u|^5 dx \right)^{2/5}$$

$$\le \sup_{0\le s\le t} |\rho|_{10/\varepsilon-5}^{1/\varepsilon - 1/2}(s) \left(\sup_{0\le s\le t} \int \rho |u|^4 dx(s) \right)^{1/2} \int_0^t |\nabla u|_5^2(s) ds$$

$$\le \Psi_\varepsilon(\int_0^t \Phi(h(s))ds) \left(\left(\int_0^t |\nabla G|_{15/8}^2 ds \right)^{9/20} + \Psi \right)$$

$$\left(\int_0^t |\nabla G|_{15/8}^2 + |\Delta \mathbf{P} u|_{15/8}^2 ds + \int_0^t \Phi(h(s))ds \right),$$

where Ψ_ε is a Lipschitz function depending on ε and we choose $\varepsilon = \frac{1}{11}$. From the estimate for $\int_0^t \int |u\nabla u|^2 dxds$, we have

$$\int_0^t \int \rho |u\nabla u|^2 \le \Psi_\varepsilon(\int_0^t \Phi(h(s))ds) \left(\int_0^t |\nabla G|_{15/8}^2 + |\Delta \mathbf{P} u|_{15/8}^2 ds \right)^{\frac{9}{10} + \frac{11\varepsilon}{20}}$$

$$+ \Psi(\int_0^t \Phi(h(s))ds).$$

Now if we choose $\varepsilon = \frac{1}{11}$, then

$$\frac{9}{10} + \frac{11\varepsilon}{20} = \frac{19}{20} < 1$$

and

$$\int_0^t \int \rho |u\nabla u|^2 \le \Psi(\int_0^t \Phi(h(s))ds) \left(\int_0^t |\nabla G|_{15/8}^2 + |\Delta \mathbf{P} u|_{15/8}^2 ds \right)^{19/20}$$

$$+ \Psi(\int_0^t \Phi(h(s))ds).$$

From integration by parts,

$$-\int \nabla P \cdot u_s dx = \int P \mathrm{div} u_s dx = \frac{d}{ds} \int P \mathrm{div} u dx + \int P_s \mathrm{div} u dx.$$

Integrating in time, we have

$$-\int_0^t \int \nabla P \cdot u_s dxds = -\int P \mathrm{div} u dx(t) + \int P_0 \mathrm{div} u_0 dx$$

$$+\int_0^t\int P'\rho_s \mathrm{div} u dxds.$$

We find that

$$\left|\int P(\rho)\mathrm{div} u dx(t)\right| \le \frac{\mu}{4}\int|\nabla u|^2dx(t)+\frac{4}{\mu}\int P^2dx$$

$$\le \frac{\mu}{4}\int|\nabla u|^2dx(t)+\Psi(\int_0^t\Phi(h(s))ds).$$

Since $\rho_t=-\mathrm{div}(\rho u)$, we find that

$$\int_0^t\int P'\rho_s\mathrm{div} u dxds = -\int_0^t\int P'\mathrm{div} u(\rho\mathrm{div} u+u\cdot\nabla\rho)dxds$$

$$-\int_0^t\int P'\rho|\mathrm{div} u|^2dxds-\int_0^t\int\nabla P\cdot u\mathrm{div} u dxds$$

$$\int_0^t\int(P-P'\rho)|\mathrm{div} u|^2dxds+\int_0^t\int Pu\cdot\nabla\mathrm{div} u dxds.$$

Clearly we have

$$\left|\int_0^t\int(P-P'\rho)|\mathrm{div} u|^2dxds\right|$$

$$\le\int_0^t|P-P'\rho|_\infty(s)|\nabla u|_2^2(s)ds$$

$$\le\int_0^t\Phi(h)ds.$$

Since

$$\mathrm{div} u=\frac{1}{\lambda+\mu}(G+P),$$

we have

$$\left|\int_0^t\int\nabla P\cdot u\mathrm{div} u dxds\right|=\frac{1}{\lambda+\mu}\left|\int_0^t\int Pu\nabla(G+P)dxds\right|$$

$$\le c\int_0^t\int P^2|\mathrm{div} u|dxds+c\int_0^t\int P|u||\nabla G|dxds$$

$$\le\left(\int_0^t|\nabla G|_{15/8}^2ds\right)^{19/20}+\Psi(\int_0^t\Phi ds).$$

Therefore combining all the estimates, we have

$$\int_0^t\int\rho|u_t|^2dxds+\int|\nabla u|^2dx(t)$$

$$\leq \Psi \left(\int_0^t |\nabla G|_{15/8}^2 + |\Delta \mathbf{P} u|_{15/8}^2 ds \right)^{19/20} + \Psi$$

$$\leq \Psi \left(\int_0^t \int \rho |u_t|^2 dx ds + \int |\nabla u|^2 dx(t) \right)^{19/20} + \Psi$$

$$\leq \frac{1}{2} \int_0^t \int \rho |u_t|^2 dx ds + \int |\nabla u|^2 dx(t) + \Psi$$

and we conclude that

$$\int_0^t \int \rho |u_t|^2 dx ds + \int |\nabla u|^2 dx(t) \leq \Psi.$$

4 L^∞-bound of density

From the mass conservation law, we have

$$(\log \rho)_t + u \cdot \nabla (\log \rho) + \mathrm{div} u = 0$$

and from momentum conservation law,

$$(\Delta^{-1} \mathrm{div}(\rho u)) + u \cdot \nabla (\Delta^{-1} \mathrm{div}(\rho u))$$

$$+ [u_j, R_i R_j](\rho u_i) - (\lambda + 2\mu) \mathrm{div} u + P = 0.$$

Thus, if we define $F = (\lambda + 2\mu) \log \rho + \Delta^{-1} \mathrm{div}(\rho u)$, F satisfies

$$F_t + u \cdot \nabla F + P = [u, , R_i R_j](\rho u_i).$$

Next we define the Lagrange flow X of u so that

$$(X(t,s,x))_t = u(t, X(t,s,x)), \quad X(s,s,x) = x$$

and derive

$$F(t, X(t,0,x)) = F_0 - \int_0^t P(\rho(s, X(s,0,x))) ds$$

$$+ \int_0^t [u, , R_i R_j](\rho u_i)(s, X(s,0,x)) ds.$$

Using the fact that ρ_0 is nonnegative, we have

$$F(t, X(t,0,x)) \leq F_0 + \int_0^t [u, , R_i R_j](\rho u_i)(s, X(s,0,x))($$

$$\log \rho(t,x) \le \log(|\rho_0|_\infty) + c|\Delta^{-1}\mathrm{div}(\rho_0 u_0)|_\infty$$
$$+c|\Delta^{-1}\mathrm{div}(\rho u)|_\infty(t) + c\int_0^t |[u,, R_iR_j](\rho u_i)|_\infty(s)ds.$$

In view of Sobolev embedding, we have

$$|\Delta^{-1}\mathrm{div}(\rho_0 u_0)|_\infty \le |\rho_0 u_0|_{7/2}$$

and

$$|\Delta^{-1}\mathrm{div}(\rho u)|_\infty(t) \le c|\rho u|_{7/2} \le |\rho|_{21}^6(t) + \left(\int \rho|u|^4 dx(t)\right)^{2/7} \le \Psi.$$

Again, from Sobolev embedding, we obtain

$$|[u,, R_iR_j](\rho u_i)|_\infty(s) \le |[u,, R_iR_j](\rho u_i)|_{W^{1,7/2}}$$
$$\le c|\nabla u|_5|\rho u|_{20} \le c|\nabla u|_5|\rho|_{39}^{39/40}|u|_\infty^{9/10}\left(\int \rho|u|^4 dx\right)^{1/40}.$$

we know that

$$|u|_\infty(s) \le |u-\overline{u}|_\infty(s) + |\overline{u}|_\infty(s) \le |\nabla u|_5(s) + \Phi(h(s))$$
$$|\rho|_{39}(s) \le \Psi(\int_0^t \Phi(h(s))ds)$$
$$\sup_{0\le s\le t}\int \rho|u|^4 dx(s) \le \Psi(\int_0^t \Phi(h(s))ds).$$

Hence, we get

$$\int_0^t |[u,, R_iR_j](\rho u_i)|_\infty(s)ds \le \int_0^t |\nabla u|_5^2(s)ds + \Psi$$
$$\le c\int_0^t |\nabla G|_{15/8}^2 + |\Delta \mathbf{P}u|_{15/8}^2 ds + \Psi \le \Psi$$

and this implies

$$\rho(t,x) \le \Psi.$$

With the estimate of $|\nabla u|_2(t)$, we conclude that

$$h(t) \le \Psi(\int_0^t \Phi(h(s))ds)$$

for some Lipschitz functions Ψ and Φ. Since Ψ and Φ are Lipschitz, there is T_0 such that

$$h(t) \le C \quad \text{for all} \quad 0 \le t \le T_0.$$

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